Metric Spaces and Topology
Lecture 18

Topological spaces

Def. A *topology* $\mathcal{T} \subseteq \mathcal{P}(X)$ on a set X is a family of subsets of X s.t.

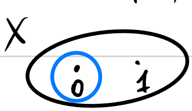
(i) $\emptyset, X \in \mathcal{T}$

(ii) $\mathcal{T}$ is closed under arbitrary (also uncbl) unions and finite intersections.

We refer to the pair $(X, \mathcal{T})$ as a *topological space*.
The sets in $\mathcal{T}$ are called *open* and their complements are called *closed*. In particular, $\emptyset, X$ are both open and closed, so clopen.

Examples. (a) For any set, the *coarsest top* on X is $\mathcal{T}_0 := \{\emptyset, X\}$, and the *finest top* on X is $\mathcal{T}_\infty := \mathcal{P}(X)$.
The top is the finest $\Leftrightarrow$ every point is open.
Coarsest top is also called *trivial*, while the finest top is also called *discrete*.

(b) $X := \{0, 1\}$ and $\mathcal{T} := \{\emptyset, X, \{0\}\}$.



$\{0\}$ is open not closed, so $\{1\}$ is closed not open. The closure (i.e. the smallest closed superset) of $\{0\}$ is X.

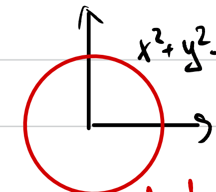
(c) For any set X, the cofinite top $\mathcal{T}$ on X is the set of all cofinite sets (i.e. complements of finite) together with $\emptyset$. This is indeed a topology since intersection of two cofinite sets is cofinite (equiv. union of two finite sets is finite).
When X is finite, this is just the discrete top.
When X is a field, this is called the Zariski top.

(d) Let $\mathbb{F}$ be an infinite field, e.g. $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and fix $n \in \mathbb{N}^+$. We turn $\mathbb{F}^n$ into a top. space by declaring the sets of solutions to (possibly infinite) systems of polynomial equations in variables $x_1, x_2, \dots, x_n$ closed (also called Zariski closed). Thus, for a polynomial $p(x_1, \dots, x_n)$, the set

$$[p] := \{ (a_1, \dots, a_n) \in \mathbb{F}^n : p(a_1, \dots, a_n) = 0 \}$$

is a closed set and so are arbitrary intersections of these.

Note that for $\mathbb{F} := \mathbb{Q}, \mathbb{R}, \mathbb{C}$, the usual Euclidean topology (given by the ℓ^2 metric) on $\mathbb{F}^n$ is finer than Zariski top because the set of roots of any polynomial is closed in the Euclidean top. (being a preimage of $0 \in \mathbb{F}$ under a continuous function).



Zariski top. is indeed a top because the closed sets are closed under arbitrary intersections, by definition. Moreover, for two polynomials $p(x_1,\dots,x_n)$ and $q(x_1,\dots,x_n)$, the union $[p] \cup [q] = [p\cdot q]$.

$\boxed{\text{HW}}$ (a) Show that Zariski closed sets are closed under finite unions.
(b) The intersection of a finitely many nonempty Zariski open sets is nonempty. Equivalently, the union of finitely many Zariski closed sets is not equal to $\mathbb{F}^n$.

Hilbert basis theorem. The set of solutions of any system of polynomials is equal to the set of solutions of a finite system of polynomials. In other words, every ideal in $F[x_1,\dots,x_n]$ is finitely generated.

All definitions for metric spaces that only used open sets remain valid for topological spaces. For example, the closure $\bar{A}$ of a subset $A \subseteq X$ is the $\subseteq$-least closed superset of A; equivalently, $\bar{A} = \{x \in X : x \text{ adheres to } A\}$, where x **adheres to** A if any open neighb. $U \ni x$ intersects A.

Def. Let X be a set and $\Sigma \subseteq \mathcal{P}(X)$ be a family of subsets. The **topology generated** by Σ is the smallest top $\mathcal{T}_\Sigma \subseteq \mathcal{P}(X)$ containing Σ. ($\mathcal{P}(X)$ is such a top and smallest exists because intersections of topologies on X is still a top. on X.)

Prop. The top $\mathcal{T}_\Sigma$ generated by Σ is the collection of all unions of finite intersections of sets in Σ together with X.

Proof. Clearly this closed under arbitrary unions by def, so it remains to show closedness under finite intersections. Let $U := \bigcup_{i \in I} U_i$ and $V := \bigcup_{j \in J} V_j$, where each U_i and V_j are finite intersections of sets from Σ. Then $U \cap V = \bigcup_{i \in I, j \in J} U_i \cap V_j$ and $U_i \cap V_j$ is still a finite intersection of sets in Σ. $\square$

For a top. $\mathcal{T}$ on X, we say that Σ is a prebasis if Σ generates $\mathcal{T}$.

A collection $\mathcal{B} \subseteq \mathcal{T}$ is called a basis for a top. $\mathcal{T}$ on X if every $\mathcal{T}$-open set is a union of sets from $\mathcal{B}$.

Cor. In a top. space $(X, \mathcal{T})$, a collection $\Sigma \subseteq \mathcal{P}(X)$ is a prebasis $\Leftrightarrow$ the collection of all finite intersections of sets in Σ is a basis for $\mathcal{T}$.

Proof. Follows from the last proposition. $\square$

For a point $x \in X$, a collection $\mathcal{B}_x \subseteq \mathcal{T}$ is called a basis at x if for every open $U \ni x$ $\exists\, V \in \mathcal{B}_x$ s.t. $V \subseteq U$.

Example. The set of all balls centred at x in a metric space forms a basis at x.